## 3.3 The Lie algebra of a Lie group
### Definition and examples

Let $G$ be a Lie group and $M$ a smooth manifold:

**Definition 3.33** [Smooth action]
A left action of $G$ on $M$ is called smooth if the action map $G \times M \longrightarrow M$ is smooth.

If $G$ acts smoothly on the left on $M$, then every $g \in G$ gives rise to a diffeomorphism

$$L_g : M \longrightarrow M.$$
$$x \longmapsto gx.$$

and hence, by Corollary 3.32 to a Lie algebra isomorphism

$$(L_g)_* : \mathrm{Vect}^\infty(M) \longrightarrow \mathrm{Vect}^\infty(M).$$

**Definition 3.34** [G-invariant vector field]
A smooth vector field $X \in \mathrm{Vect}^\infty(M)$

is <u>$G$-invariant</u> if $\forall g \in G \ (L_g)_* X = X$.

**Definition 3.35** [Lie subalgebra]

Let $\mathfrak{g}$ be a Lie algebra. A vector subspace $\mathfrak{h} \subset \mathfrak{g}$ is a <u>Lie subalgebra</u> if $[X, Y] \in \mathfrak{h}$ whenever $X, Y \in \mathfrak{h}$.

By Corollary 3.32 the subvector space $\mathrm{Vect}^\infty(M)^G$ of <u>$G$-invariant vector fields</u> in $\mathrm{Vect}^\infty(M)$ is a <u>Lie subalgebra</u> of $\mathrm{Vect}^\infty(M)$.

Let now $G$ act on the left <u>on itself</u>.

$$G \times G \longrightarrow G$$
$$(g, x) \longmapsto g \cdot x$$

Then $\mathrm{Vect}^\infty_L(G)^G$ the space of <u>left invariant vector fields</u> is a Lie algebra.

Moreover we have the following:

**Lemma 3.36**

$$\mathrm{Vect}^\infty_L(G)^G \longrightarrow T_e G$$
$$X \longmapsto X_e.$$

is a vector space <u>isomorphism</u>.

{ In particular the Lie algebra of left-invariant vector fields is finite dimensional

**Proof**

We define a map $T_e G \longrightarrow \mathrm{Vect}^\infty_L(G)$
$v \longmapsto v^L$

as follows

$$v^L_g := D_e L_g (v) \in T_g G.$$

The fact that $v^L \in \mathrm{Vect}^\infty_L(G)^G$ follows from the <u>chain rule</u>. hence $D_e L_e = \mathrm{id}$

Note also that $v^L_e = v$ since $L_e = \mathrm{id}_G$.

On the other hand, if $X \in \mathrm{Vect}^\infty_L(G)^G$ then in particular:

left-invariance

$$X_g = (D_e L_g).(X_e)$$

and hence. $X = (X_e)^L$. $\square$

We are ready to introduce the definition of <u>Lie algebra</u> of a <u>Lie group</u> :

**Definition 3.37** [Lie algebra of a Lie group]
The <u>Lie algebra</u> $\mathfrak{g}$ of a Lie group $G$ is the vector space $\mathfrak{g} = T_e G$ endowed with the <u>bracket</u> $[v,w] = [v^L, w^L]_e$, $\forall\, v,w \in T_e G$

—— o ——

We would like to identify explicitly the Lie algebra of $GL(n,\mathbb{R})$. Recall that since $GL(n,\mathbb{R}) \subset M_{n,n}(\mathbb{R})$ is open we have the identification

$$\begin{aligned} M_{n,n}(\mathbb{R}) &\longrightarrow T_I\, GL(n,\mathbb{R}) \\ A &\longmapsto A_I . \end{aligned}$$

Let us denote $\mathfrak{gl}(n,\mathbb{R})$ the Lie algebra of $GL(n,\mathbb{R})$ and for convenience

$$\tilde{A} = (A_I)^L$$

the left-invariant vector field corresponding to $A_I$.

Then we have:

**Proposition 3.38**

The map

$$M_{n,n}(\mathbb{R}) \longrightarrow \mathfrak{gl}(n,\mathbb{R})$$
$$A \longmapsto \tilde{A}$$

induces an <u>isomorphism</u> between the Lie algebra $M_{n,n}(\mathbb{R})$ with <u>matrix bracket</u> and the Lie algebra $\mathfrak{gl}(n,\mathbb{R})$. Equivalently

$$[\tilde{A}, \tilde{B}] = \widetilde{[A,B]} \quad \forall A, B \in M_{n,n}(\mathbb{R})$$

⇝ <u>End of Lecture</u>

<u>Proof</u>

Since both $[\tilde{A}, \tilde{B}]$ and $\widetilde{[A,B]}$ are left-invariant vector fields it suffices to check that:

$$[\tilde{A}, \tilde{B}]_I = \widetilde{[A,B]}_I \,.$$

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By the identifications that we discussed a few pages ago two tangent vectors coincide iff their evaluation on all $\lambda \in M_{n,n}(\mathbb{R})^*$ do.

Therefore it is sufficient to show

$$[\tilde{A},\tilde{B}]_I(\lambda) = \widetilde{[A,B]}_I(\lambda)$$

$\forall \lambda \in M_{n\times n}(\mathbb{R})^*$.

Note that $\widetilde{[A,B]}_I = [A,B]$.

Hence recalling also that linear functions can be identified with their derivatives we need to show that

Cf. See page 26-27

$$\lambda([A,B]) = [\tilde{A},\tilde{B}]_I(\lambda).$$

However $\lambda([A,B]) = \lambda(AB) - \lambda(BA)$ since $[A,B]$ is the bracket in $M_{n\times n}(\mathbb{R})$.

On the other hand:

$$[\tilde{A},\tilde{B}]_I(\lambda) = (\tilde{A}\tilde{B} - \tilde{B}\tilde{A})(\lambda)(I)$$

We proceed to show that:

$$\tilde{A}\tilde{B}(\lambda)(I) = \lambda(AB).$$

This will be enough to complete the proof.
~ the other term can be treated analogously

$$\tilde{A}\tilde{B}(\lambda)(I) = A_I(\tilde{B}(\lambda)).$$

$\tilde{B}$ is left invariant $= A_I(g \longmapsto \tilde{B}_g(\lambda))$

$$= A_I(g \longmapsto D_I L_g(B_I).(\lambda)).$$

Definition of differential

But $D_I L_g(B_I)(\lambda) = B_I(h \longmapsto \lambda(gh))$

Furthermore. $h \longmapsto \lambda(gh)$. is the restriction of a linear form on $M_{n,n}(\mathbb{R})$ to $GL(n,\mathbb{R})$

~ usual identifications

Hence. $B_I(\lambda \longmapsto \lambda(gh)) = \lambda(gB)$.

Hence $\tilde{A}\tilde{B}(\lambda)(I) = A_I(g \longmapsto \lambda(gB))$
and for the same reasons as above.

$$A_I(g \longmapsto \lambda(gB)) = \lambda(AB) \text{ as claimed.}$$

□

Our next goal will be to understand whether a <u>smooth homomorphism of Lie</u>.

groups induces a Lie algebra homomorphism.

We have the following:

**Proposition 3.39**
Let $\varphi : G \longrightarrow H$ be a smooth homomorphism of Lie groups and $\mathfrak{g} = T_e G$ and $\mathfrak{h} = T_e H$ be their Lie algebras. Then $D_e\varphi : \mathfrak{g} \longrightarrow \mathfrak{h}$ is a Lie algebra homomorphism.

Proof

Let $v \in T_e G$, $v^L \in \mathrm{Vect}^\infty(G)^G$ the corresponding left invariant vector field. $w := D_e\varphi(v) \in T_e H$ and $w^L \in \mathrm{Vect}^\infty(H)^H$

Claim: $v^L$ and $w^L$ are $\varphi$-related.

To prove the claim we note that

$$w^L_{\varphi(g)} \overset{\text{def of } w^L}{=} D_e L_{\varphi(g)}(w) \overset{\text{def of } w}{=} D_e L_{\varphi(g)} D_e\varphi(v)$$

$$= D_e\left(L_{\varphi(g)}\circ\varphi\right)(v).$$

Chain rule

Since $\varphi$ is a homomorphism

$$L_{\varphi(g)}\circ\varphi = \varphi\circ L_g,$$

hence.

Chain rule again.

$$= D_e\left(\varphi\circ L_g\right)(v) = D_g\varphi\left(D_e L_g(v)\right)$$

$$= D_g\varphi\left(v_g^L\right).$$

Def $v^L$

Thus if $v_1, v_2 \in T_e G$ and $w_i := D_e\varphi(v_i)$ then since $v_i^L$ and $w_i^L$ are $\varphi$-related, it follows from Proposition 3.3.1 that $[v_1^L, v_2^L]$ and $[w_1^L, w_2^L]$ are $\varphi$-related.

Hence

def of $[\,,\,]$ on the Lie algebra

$$D_e\varphi([v_1,v_2]) = D_e\varphi\left([v_1^L, v_2^L]_e\right)$$

$\varphi$-related

$$= [w_1^L, w_2^L]_e.$$

Definition of bracket on $\mathfrak{h}$

$$= [w_1, w_2]$$

Definition of $w_e'$ $\quad = [D_e\varphi(v_1), D_e\varphi(v_2)]$. $\square$

**Corollary 3.40**

Let $G$ be a Lie group and $H < G$ be a subgroup which is also a regular submanifold. Then the inclusion $H \longrightarrow G$ realizes $\mathfrak{h} = T_e H$ as a Lie subalgebra of $\mathfrak{g} = T_e G$.

**Example 3.41**

1) The Lie algebra of $SL(n,\mathbb{R})$ is:

$$sl(n,\mathbb{R}) = \{ X \in M_{n,n}(\mathbb{R}) : \operatorname{tr} X = 0 \}$$

(cf with Example 3.14 1)).

Indeed we saw that $SL(n,\mathbb{R}) = \det^{-1}(1)$ and $\det : GL(n,\mathbb{R}) \longrightarrow \mathbb{R}^*$ has constant rank with $(D_I \det)(X) = \operatorname{tr} X$

Therefore if $\gamma : (-\varepsilon, \varepsilon) \longrightarrow SL(n,\mathbb{R})$ is a smooth curve $\frac{d}{dt}\det(\gamma(t)) = 0$.

If we choose $\gamma$ such that $\gamma(0) = 1$ and so.

$\gamma'(0) \in T_I\, SL(n,\mathbb{R}) = sl(n,\mathbb{R})$ then

$$0 = \frac{d}{dt}\Big|_{t=0} \det(\gamma(t)) = D_I \det(\gamma'(0))$$

Have a look at [Lee, page 68-69 ....]

$$= tr(\gamma'(0))$$

Hence $sl(n,\mathbb{R}) \subseteq \{A \in gl(n,\mathbb{R}) : tr A = 0\}$

for every tangent vector y can find a smooth curve with that vector as speed.

Since $\dim sl(n,\mathbb{R}) = \dim \{A \in gl(n,\mathbb{R}) : tr A = 0\}$

Both $= n^2 - 1$.

equality in the above inclusion holds.

2). The Lie algebra of $O(n,\mathbb{R})$ is
$o(n,\mathbb{R}) = \{X \in M_{n,n}(\mathbb{R}) : X + {}^tX = 0\}$
( cf with Example 3.14 2) ).

For checking the above it is helpful to keep in mind the following:

If $A, B : (-\varepsilon, \varepsilon) \longrightarrow M_{n,n}(\mathbb{R})$ are smooth curves and we set
$p(s) := A(s) \cdot B(s) \in M_{n,n}(\mathbb{R})$

then $p$ is a smooth curve and

$$p'(s) = A'(s)B(s) + A(s)B'(s).$$

3) Note that $N = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}$ is a subgroup and a regular submanifold of $GL(n,\mathbb{R})$. Its Lie algebra is

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \right\}.$$

Analogously for $A = \left\{ \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} : \lambda_i \in \mathbb{R}_{>0} \right\}$

we have

$$\mathfrak{a} = \left\{ \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix} : x_i \in \mathbb{R} \right\}$$

Note that $[\,,\,]$ vanishes on $\mathfrak{a}$.

$\boxed{\text{Exercise 3.42.}}$

1) Compute the Lie algebra of $O(p,q)$ and $SO(p,q)$ for $p+q=n$.

2) Realise. $GL(n,\mathbb{C})$, $SL(n,\mathbb{C})$, $U(n)$ as Lie groups and compute their Lie algebras.

## Example 3.43

Let $G, H$ be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Then the Lie algebra of $G \times H$ can be identified with $\mathfrak{g} \times \mathfrak{h}$ with bracket

$$[(x_1,y_1),(x_2,y_2)] := ([x_1,x_2]_{\mathfrak{g}}, [y_1,y_2]_{\mathfrak{h}}).$$

(Exercise).

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In abstract terms Proposition 3.39 says that we have constructed a function

$$\text{Lie}: \text{Lie groups} \longrightarrow \text{Lie algebras}.$$
$$G \qquad\qquad\qquad \mathfrak{g}$$

(a "morphism" (= Lie group homomorphism) between Lie groups naturally induces a morphism (= Lie algebra homomorphism) between the respective Lie algebras).

The fundamental question is how much information we lose by going from Lie groups to Lie algebras. (+ additional properties.)

Some informal remarks follow. We will clarify some of them over the next few lectures.

(Finite dimensional)

1) Every Lie algebra is the Lie algebra of a Lie group. More generally we shall discuss the "Lie group - Lie algebra correspondence".

2) [Faithfulness] Note that if $G$ is a Lie group and $F$ is any finite group with the discrete topology then $G$ and $G \times F$ have the same Lie algebra.

It might seem that this is related to disconnectedness. However, even if $G$ is connected, it is not uniquely determined by its Lie algebra.

For instance. we note that:

$\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2/\mathbb{Z}^2$ is a covering map and it is easy to see that it induces an isomorphism between the invariant vector fields and hence between the Lie algebras.

In fact if $G_1$ and $G_2$ are connected Lie groups then any isomorphism,

$$\mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$$

comes from an isomorphism $\tilde{G}_1 \to \tilde{G}_2$ (we will probably prove this later.).

3). It would be nice if the category of Lie groups is closed under certain <u>natural operations</u> like taking the <u>center</u> $Z(G)$ of a Lie group $G$, or $G^0$ <u>the connected component</u> of the identity,

In this direction we will see a very important theorem due to Cartan saying that if $H < G$ <u>is a closed</u>

subgroup then it is a regular submanifold and hence a Lie group.